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# The mesonic chiral lagrangean of order $p^6$ *

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Abstract: We construct the effective chiral lagrangean for chiral perturbation theory in the mesonic even-intrinsic-parity sector at order $p^6$. The lagrangean contains 112 in principle measurable + 3 contact terms for the general case of $n$ light flavours, 90+4 for three and 53+4 for two flavours. The equivalence between equations of motion and field redefinitions to remove spurious terms in the lagrangeans is shown to all orders in the chiral expansion. We also discuss and implement other methods for reducing the number of terms to a minimal set.

Keywords: Nonperturbative Effects, Spontaneous Symmetry Breaking, QCD, Chiral Lagrangeans.

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## Contents





# 1. Introduction

The low-energy limit of the theory of the strong interaction, QCD, is chiral perturbation theory (CHPT). This is the effective field theory method of solving in a long-distance expansion the Ward identities of the chiral symmetry of QCD. CHPT in the meson sector [1, 2, 3][1] is now being carried out at next-to-next-to-leading order. By now, several complete calculations to $O(p^6)$ exist [5, 6] and the renormalization of the generating functional of Green functions of quark currents will soon be available to this order [7]. The double divergences proportional to $1/(d-4)^2$ are already known [8].

In each specific calculation, the local contribution of $O(p^6)$ is a rather trivial part when compared to the two-loop contributions. In those calculations, no attempt is usually made to relate the low-energy constants of $O(p^6)$ in the local amplitudes to those appearing in other processes. This is precisely the purpose of the present paper, to establish the most general local solution of chiral Ward identities at the level of the generating functional. This solution amounts to constructing the effective chiral lagrangean of $O(p^6)$, which is also invariant under Lorentz transformations, parity ($P$) and charge conjugation ($C$). We use here the external field method of [2, 3] where only terms explicitly invariant under these symmetries are relevant.

[1] An overview of review articles and lectures as well as recent results can be found in [4].

It is relatively easy to write down such chiral lagrangeans. The real challenge is, of course, to find the minimal set of terms for those lagrangeans. The corresponding low-energy constants then parametrize the most general local solutions of the chiral Ward identities. We construct this lagrangean first for a general number $n$ of light flavours and then specialize to the phenomenologically relevant cases $n = 2$ and 3 where the number of independent terms is substantially smaller. We confine ourselves to the lagrangeans of even intrinsic parity, i.e. to terms without an $\varepsilon$ tensor. We also compare our results with the work of Fearing and Scherer [9] who have previously published chiral lagrangeans of $O(p^6)$ for general $n$ and for $n = 3$.

In constructing the chiral lagrangean of $O(p^6)$, we use partial integration (in the corresponding action) and the equations of motion (EOM) of the lowest-order lagrangean to reduce the number of chiral invariants. To make the large-$N_c$ counting transparent, where $N_c$ is the number of colours, we employ the various relations among different monomials of $O(p^6)$ to eliminate preferentially terms with multiple flavour traces.[2] The final lagrangeans are ordered essentially according to the external fields in their components.

The paper is organized as follows. In sect. 2, we start by collecting the ingredients for constructing the effective chiral lagrangean of $O(p^6)$ for chiral SU($n$). We use a basis where all (matrix) operators have the same chiral transformation properties. In addition to partial integration and the EOM, we use the Bianchi identity for the field strength tensor on chiral coset space to reduce the lagrangean to a minimal form. We also extract the so-called contact terms that depend only on external fields. In sect. 3, we simplify the lagrangean further for chiral SU(3) and finally for SU(2). It turns out that all 21 linear relations for $n = 3$ as well as the additional 37 relations for $n = 2$ can be derived from the respective Cayley-Hamilton relations. We discuss possible applications and limitations of the chiral lagrangeans of $O(p^6)$ in sect. 4. Sect. 5 contains some conclusions. In app. A we present an explicit proof for the equivalence between using the EOM and field transformations for simplifying the lagrangeans. App. B contains the complete list of linear relations of the Cayley-Hamilton type for both $n = 2$ and 3. Finally, we compare with the chiral lagrangeans of Fearing and Scherer [9] in app. C and demonstrate with some explicit examples why we end up with substantially fewer terms.

## 2. Chiral SU($n$)

In the formulation of Gasser and Leutwyler [2, 3], the QCD lagrangean $\mathcal{L}^0_{\rm QCD}$ with $n$ massless quarks is enlarged to

$$\mathcal{L} = \mathcal{L}^0_{\rm QCD} + \overline{q}\gamma^\mu(v_\mu + a_\mu\gamma_5)q - \overline{q}(s - ip\gamma_5)q \tag{2.1}$$

[2] In the limit of large $N_c$, terms with single flavour traces dominate. Each additional flavour trace brings in a suppression of order $1/N_c$ [10].

by coupling the quarks to external $n \times n$-dimensional matrix fields in flavour space, $v_\mu, a_\mu, s, p$. At the quantum level, the theory exhibits a (local) chiral symmetry $\mathrm{SU}(n)_L \times SU(n)_R \times \mathrm{U}(1)_V$ that is spontaneously broken to $\mathrm{SU}(n)_V \times \mathrm{U}(1)_V$.

The basic building block of chiral lagrangeans is the Goldstone matrix field $u(\varphi)$ transforming as

$$u(\varphi) \to u(\varphi') = g_R u(\varphi) h(g,\varphi)^{-1} = h(g,\varphi) u(\varphi) g_L^{-1} \tag{2.2}$$

under a general chiral rotation $g = (g_L, g_R) \in \mathrm{SU}(n)_L \times \mathrm{SU}(n)_R$ in terms of the compensator field $h(g,\varphi)$. Mesonic chiral lagrangeans can be constructed by taking (products of) traces of products of chiral operators $X$ that either transform as

$$X \to h(g,\varphi) X h(g,\varphi)^{-1} \tag{2.3}$$

or remain invariant under chiral transformations. The simplest such operators are

$$\begin{aligned} u_\mu &= i\{u^\dagger(\partial_\mu - i r_\mu)u - u(\partial_\mu - i\ell_\mu)u^\dagger\}\,, \\ \chi_\pm &= u^\dagger \chi u^\dagger \pm u \chi^\dagger u\,, \end{aligned} \tag{2.4}$$

with $r_\mu = v_\mu + a_\mu$, $\ell_\mu = v_\mu - a_\mu$, $\chi = 2B(s + ip)$. The mesonic chiral lagrangean of lowest order can be written as

$$\mathcal{L}_2 = \frac{F^2}{4} \langle u_\mu u^\mu + \chi_+ \rangle\,, \tag{2.5}$$

with two low-energy parameters $F, B$ and with the usual notation $\langle \ldots \rangle$ for the flavour trace.

In higher orders, we need additional operators for the construction of chiral lagrangeans. Up to and including $O(p^6)$, the following operators are sufficient for this purpose:

$$\begin{aligned} f_\pm^{\mu\nu} &= u F_L^{\mu\nu} u^\dagger \pm u^\dagger F_R^{\mu\nu} u\,, \qquad \nabla_\lambda f_\pm^{\mu\nu}\,, \\ h_{\mu\nu} &= \nabla_\mu u_\nu + \nabla_\nu u_\mu\,, \\ \chi_{\pm\mu} &= u^\dagger D_\mu \chi u^\dagger \pm u D_\mu \chi^\dagger u = \nabla_\mu \chi_\pm - \frac{i}{2}\{\chi_\mp, u_\mu\}\,, \end{aligned} \tag{2.6}$$

with non-Abelian field strengths

$$\begin{aligned} F_R^{\mu\nu} &= \partial^\mu r^\nu - \partial^\nu r^\mu - i[r^\mu, r^\nu]\,, \\ F_L^{\mu\nu} &= \partial^\mu \ell^\nu - \partial^\nu \ell^\mu - i[\ell^\mu, \ell^\nu] \end{aligned} \tag{2.7}$$

and with $D_\mu \chi = \partial_\mu \chi - i r_\mu \chi + i \chi l_\mu$. The covariant derivative

$$\nabla_\mu X = \partial_\mu X + [\Gamma_\mu, X] \tag{2.8}$$

is defined in terms of the chiral connection

$$\Gamma_\mu = \frac{1}{2}\{u^\dagger(\partial_\mu - i r_\mu)u + u(\partial_\mu - i\ell_\mu)u^\dagger\}. \tag{2.9}$$

With traceless matrix gauge fields $v_\mu$, $a_\mu$, the matrices $u_\mu$, $f_\pm^{\mu\nu}$, $h_{\mu\nu}$ and $\nabla_\lambda f_\pm^{\mu\nu}$ are also traceless. The antisymmetric combination $\nabla_{[\mu} u_{\nu]}$ is not an independent quantity because of the identity

$$f_-^{\mu\nu} = \nabla^\nu u^\mu - \nabla^\mu u^\nu \,. \tag{2.10}$$

We use $f_-^{\mu\nu}$ and $\chi_\pm^\mu$ in order to have as few terms as possible for vanishing external fields.

An equivalent basis is the L(eft)R(ight)-basis with $U = u^2$ in the conventional choice of coset coordinates. We have found it more convenient to use operators transforming as in (2.3) for at least two reasons:

- functional integration produces functionals of such fields in a natural way, in particular the divergent parts [3, 7, 8];
- one of the main tasks in constructing the lagrangean of $O(p^6)$ is to find a minimal set closed under partial integration. This is easier to achieve with operators of type (2.3) than in the LR-basis [9].[3]

The construction of mesonic chiral lagrangeans proceeds by writing down all Lorentz invariant (products of) traces of products of chiral operators $X$ with the required chiral dimension. This guarantees chiral symmetry, but one has to implement in addition the discrete symmetries $P$ and $C$ and hermiticity of the lagrangean. This is straightforward with the help of the transformation properties given in table 1.

| operator | $P$ | $C$ | h.c. |
|---|---|---|---|
| $u_\mu$ | $-\varepsilon(\mu) u_\mu$ | $u_\mu^T$ | $u_\mu$ |
| $h_{\mu\nu}$ | $-\varepsilon(\mu)\varepsilon(\nu) h_{\mu\nu}$ | $h_{\mu\nu}^T$ | $h_{\mu\nu}$ |
| $\chi_\pm$ | $\pm\chi_\pm$ | $\chi_\pm^T$ | $\pm\chi_\pm$ |
| $f_\pm^{\mu\nu}$ | $\pm\varepsilon(\mu)\varepsilon(\nu) f_\pm^{\mu\nu}$ | $\mp f_\pm^{\mu\nu T}$ | $f_\pm^{\mu\nu}$ |

**Table 1:** $P$, $C$ and hermiticity properties of operators contained in chiral lagrangeans. Space-time arguments are suppressed and we do not list the derivatives $\chi_{\pm\mu}$ , $\nabla_\lambda f_\pm^{\mu\nu}$ separately. $\varepsilon(0) = -\varepsilon(\mu \neq 0) = 1$.

There is a very large number of terms with chiral dimension six that fulfill all symmetry constraints. However, many of those terms are linearly dependent. To obtain a minimal set of independent monomials of $O(p^6)$ for chiral SU($n$), we use the following relations or procedures:

1. partial integration in the chiral action of $O(p^6)$;
2. EOM for the lowest-order lagrangean (2.5);
3. Bianchi identity;
4. contact terms.

[3] To be more precise, a right-right basis was used in ref. [9]. In principle, the infinities can also be calculated directly in this basis [11] but we have used the standard method in [7, 8].

Before discussing these simplifications in more detail, we present in table 2 the complete list of independent monomials of $O(p^6)$ for chiral SU($n$). There are 112 such terms plus three contact terms that depend only on external fields. We have ordered the monomials by introducing subsequently operators containing external fields $\chi_+$, $\chi_-$, $f_+^{\mu\nu}$ and $f_-^{\mu\nu}$, in this order. For practical purposes, there is one exception to this rule: the terms with six powers of the vielbein field $u_\mu$ are listed after the introduction of $\chi_+$ and $\chi_-$, but before terms involving $f_+^{\mu\nu}$. Such terms will only be relevant for experimentally rather remote processes involving, e.g., six mesons.

As a check for the completeness and linear independence of the 112 terms in table 2, we have also employed a different basis with higher covariant derivatives that occur naturally in the calculation [7, 8] of the divergence functional of $O(p^6)$. We have explicitly constructed the linear transformation that transforms the two bases into one another. Table 2 also contains the lagrangeans for $n = 2$ and 3 where additional relations exist, as we will discuss in the next section.

The main tools for reducing the lagrangean of $O(p^6)$ to its minimal form are partial integration in the action and the EOM. Although straightforward in principle, it is in practice nontrivial with the huge number of possible monomials to find a minimal set closed under partial integration. In particular, partial integration together with the EOM allow to reduce all higher-derivative terms to monomials involving at most single-derivative operators. It is not very illuminating to write down all possible relations of this type but we will demonstrate the procedure with some explicit examples in app. C. In fact, the optimal use of partial integration is one of the reasons why we arrive at a smaller number of independent terms than ref. [9]. With partial integration and application of the EOM, there are still 117 seemingly independent monomials of $O(p^6)$.

The loop expansion can be viewed as an expansion around the classical solution, i.e. the solution of the EOM. For a systematic chiral counting, one expands around the solution of the EOM of the lowest-order lagrangean (2.5):

$$\nabla^\mu u_\mu = \frac{i}{2}\left(\chi_- - \frac{1}{n}\langle\chi_-\rangle\right). \tag{2.11}$$

The generating functional of $O(p^6)$ contains the action for the lagrangean of $O(p^6)$ precisely at the classical solution. We may therefore replace all occurrences of $\nabla^\mu u_\mu$ or $h^\mu_\mu$ by the EOM. As is known to many practitioners in quantum field theory, application of the EOM is equivalent to field transformations, in our case of the matrix field $u(\varphi)$. We include a general proof of this equivalence in app. A.

The chiral coset space has a geometric structure encoded in the connection (2.9). The field strength tensor $\Gamma_{\mu\nu}$ associated with this connection is given by

$$\begin{aligned}[\nabla_\mu, \nabla_\nu]\, X &= [\Gamma_{\mu\nu}, X] \\ \Gamma_{\mu\nu} &= \frac{1}{4}[u_\mu, u_\nu] - \frac{i}{2} f_{+\mu\nu}\end{aligned} \tag{2.12}$$

and obeys the Bianchi identity

$$\nabla_\mu \Gamma_{\nu\rho} + \nabla_\nu \Gamma_{\rho\mu} + \nabla_\rho \Gamma_{\mu\nu} = 0 \,. \tag{2.13}$$

Since the Bianchi identity is of $O(p^3)$, it will lead to relations among $p^6$ monomials when traced with additional chiral operators of chiral dimension three. Moreover, $\nabla_\rho \Gamma_{\mu\nu}$ is a third-rank Lorentz tensor of even intrinsic parity. This implies that the Bianchi identity can only give rise to nontrivial relations when traced with either $h_{\mu\nu} u_\rho$, $f_{-\mu\nu} u_\rho$ or $\nabla_\rho f_{+\mu\nu}$. It turns out that there are only two independent relations among the 117 monomials for general $n$ when partial integration and the EOM are applied. We eliminate in this way the following two monomials that would otherwise appear in the general list of table 2:

$$\langle f_{+\mu\nu} [\chi_-^\mu, u^\nu] \rangle \,, \qquad \langle \nabla^\mu f_{+\mu\nu} \nabla_\rho f_+^{\rho\nu} \rangle \,. \tag{2.14}$$

This reduces the number of terms to 115 independent ones.

Finally, we turn to the contact terms. Our basis of operators tends to conceal the fact that some combinations depend only on external fields and are therefore not directly accessible experimentally. It is easier to express those contact terms in the LR-basis. For general $n$, there are three independent contact terms listed in table 2 as entries 113, 114 and 115 in the SU($n$) column. The covariant derivative $D$ contains only external gauge fields depending on which object it acts. For the derivation, we again used partial integration and also the Bianchi identities for $F_L^{\mu\nu}$ and $F_R^{\mu\nu}$ alone. Of course, these three contact terms can be written as linear combinations of $O(p^6)$ monomials in our basis. We eliminate the following three monomials in terms of the contact terms and of other terms contained in the SU($n$) list of table 2:

$$\begin{aligned}
\langle \chi_{-\mu} \chi_-^\mu \rangle &= Y_{47} - 4\, Y_{113} \,, \\
i \langle f_{+\mu\nu} [f_+^{\nu\rho}, f_{+\rho}^\mu] \rangle &= -3\, Y_{101} - 8\, Y_{114} \,, \\
\langle \nabla_\rho f_{+\mu\nu} \nabla^\rho f_+^{\mu\nu} \rangle &= 3/2\, Y_{71} - 3/2\, Y_{73} - 4\, Y_{75} + 2\, Y_{78} - \\
&\quad -1/2\, Y_{90} + 1/2\, Y_{92} - 2\, Y_{100} + 2\, Y_{101} + \\
&\quad +1/2\, Y_{104} - \; Y_{109} - 4\, Y_{111} + 2\, Y_{115} \,.
\end{aligned} \tag{2.15}$$

Here and in the following, $Y_i$ stands for the $i$-th monomial in the SU($n$) column of table 2.

## 3. Chiral lagrangeans for $n = 2, 3$

The chiral lagrangean of $O(p^6)$ contains 112 independent terms plus three contact terms for general $n$. Of course, for phenomenological applications only $n = 2$ and 3 are directly relevant.



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For $n = 3$, there are additional linear relations among the invariants of $O(p^6)$ due to the Cayley-Hamilton theorem whereby any $n$-dimensional matrix obeys its own characteristic equation. For $n = 3$, this relation implies the following identity among three arbitrary three-dimensional matrices $A, B, C$:

$$\begin{aligned}
&ABC + ACB + BAC + BCA + CAB + CBA - AB\langle C\rangle - \\
&\quad - AC\langle B\rangle - BA\langle C\rangle - BC\langle A\rangle - CA\langle B\rangle - CB\langle A\rangle - A\langle BC\rangle - \\
&\quad - B\langle AC\rangle - C\langle AB\rangle - \langle ABC\rangle - \langle ACB\rangle + A\langle B\rangle\langle C\rangle + B\langle A\rangle\langle C\rangle + \\
&\quad + C\langle A\rangle\langle B\rangle + \langle A\rangle\langle BC\rangle + \langle B\rangle\langle AC\rangle + \langle C\rangle\langle AB\rangle - \langle A\rangle\langle B\rangle\langle C\rangle = 0\,. \qquad (3.1)
\end{aligned}$$

Careful analysis leads to 21 independent relations of the Cayley-Hamilton variety. Thus, we can dispose of 21 of the SU($n$) monomials for $n = 3$. The explicit relations are reproduced in app. B. We have chosen to eliminate preferentially terms with multiple traces to have a transparent large-$N_c$ counting in the final $O(p^6)$ lagrangean.

We are not aware of a general proof that the Cayley-Hamilton relations exhaust all possible linear relations among traces of products of three-dimensional matrices. Moreover, our matrices have in general special properties such as being hermitian and/or traceless. We have therefore investigated whether there could be additional linear relations among the 94 monomials listed in table 2 for chiral SU(3). In other words, we have looked for nontrivial solutions of the linear equation

$$\sum_{i=1}^{94} x_i O_i = 0\,, \qquad (3.2)$$

where the $O_i$ denote the 94 monomials relevant for $n = 3$. Of course, eq. (3.2) decomposes into several independent equations with identical building blocks in the $O_i$. For example, the monomials $O_{40}, \ldots, O_{47}$ can be written as sums of scalar products of power six of eight real vectors $u^a_\mu$ ($a = 1, \ldots, 8$) with $u_\mu = u^a_\mu \lambda^a$. Without further restrictions on these vectors, $O_{40}, \ldots, O_{47}$ turn out to be linearly independent. In general the result of the analysis is that there are indeed no additional relations for $n = 3$ beyond Cayley-Hamilton. In other words, the unique solution of (3.2) is $x_i = 0$ for all $i = 1, \ldots, 94$.

One immediate bonus of this analysis is that we obtain all additional linear relations for chiral SU(2) by restricting eq. (3.2) to two dimensions. The resulting 37 independent relations are listed in app. B. They can all be interpreted as consequences of the two-dimensional Cayley-Hamilton theorem which implies the relation

$$\{A, B\} = A\langle B\rangle + B\langle A\rangle + \langle AB\rangle - \langle A\rangle\langle B\rangle\,, \qquad (3.3)$$

for arbitrary two-dimensional matrices $A, B$. The most direct way to verify those relations is of course by making use of the algebra of Pauli matrices.

In addition to the three contact terms for general $n$, there is a kind of contact term which shows up at different chiral orders, depending on $n$. This is[4] $\det(\chi)+\det(\chi^\dagger)$, which is a term of order $p^{2n}$. This appears at $O(p^4)$ for SU(2) and at $O(p^6)$ for SU(3). In the latter case we can express it in our basis as follows:

$$\det(\chi)+\det(\chi^\dagger) = \frac{1}{12}Y_{25} - \frac{1}{8}Y_{26} + \frac{1}{24}Y_{27} + \frac{1}{4}Y_{39} - \frac{1}{8}Y_{40} - \frac{1}{4}Y_{41} + \frac{1}{8}Y_{42} \ . \quad (3.4)$$

We choose to trade $Y_{42}$ for the contact term (3.4) in the SU(3) basis given in table 2, bringing the final count to 90 independent terms $O_i$ plus 4 contact terms for chiral SU(3).

The equivalent of the above term for $n=2$ shows up at $O(p^4)$. The presence of this contact term can also be explained by the fact that all representations of SU(2) are self-conjugate: $\widetilde{\chi} = \tau_2\chi^T\tau_2$ transforms like $\chi^\dagger$ under chiral SU(2) transformations, and $\det(\chi) = 1/2\langle\chi\widetilde{\chi}\rangle$. Therefore we can also construct a new contact term at $O(p^6)$ for $n=2$ by inserting derivatives: $\langle D_\mu\chi D^\mu\widetilde{\chi}\rangle$ + h.c. which is listed at the end of table 2. We can trade one more invariant for this contact term by using the relation:

$$\langle\chi_{-\mu}\rangle\langle\chi_-^\mu\rangle = 2Y_{47} - Y_{48} - 4Y_{113} + 2\left(\langle D_\mu\chi D^\mu\widetilde{\chi}\rangle + \text{ h.c.}\right) , \quad (3.5)$$

bringing the final count to 53 independent terms $P_i$ plus 4 contact terms for chiral SU(2).

The mesonic chiral lagrangeans of $O(p^6)$ then take the following final forms:

$$\begin{aligned} \mathcal{L}_6^{\mathrm{SU}(3)} &= \sum_{i=1}^{90} C_i O_i + 4 \text{ contact terms}\,, \\ \mathcal{L}_6^{\mathrm{SU}(2)} &= \sum_{i=1}^{53} c_i P_i + 4 \text{ contact terms}\,, \end{aligned} \quad (3.6)$$

with 90 (53) low-energy constants $C_i$ ($c_i$) for $n=3$ (2). These low-energy constants parametrize the most general local solutions of the chiral Ward identities. Most of them have a divergent part to cancel the divergences [7, 8] of the one- and two-loop functionals of $O(p^6)$. The scale-dependent remainders $C_i^r(\mu)$ $[c_i^r(\mu)]$ are at least in principle all measurable quantities.

## 4. A guided tour through $\mathcal{L}_6$

The number of new constants appearing at order $p^6$ is very large. One cannot hope to determine all of them from experiments, as could be done with few additional assumptions for $\mathcal{L}_4$. On the other hand, a closer look at the various terms appearing

[4]We are indebted to Bachir Moussallam for pointing out that we had overlooked this contact term for $n=3$ in the original version of the paper.



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in $\mathcal{L}_6$ shows that a large fraction of them is not phenomenologically relevant. We allude here to terms like $Y_{49}, \ldots, Y_{63}$ that contribute to processes involving at least six mesons, or like $Y_{101}$ that involves at least one vector and two axial currents. To help the reader in identifying such terms, the last column in table 2 shows the simplest quantity or process to which each term contributes.

Actually, the situation is not as hopeless as it may look at first sight. The number of terms relevant for phenomenological purposes is still manageable. To better illustrate this claim, we concentrate here on the lagrangean for $n = 2$ and in the chiral limit, for $s = p = 0$ and $v_\mu, a_\mu \neq 0$. This is practically all one needs for phenomenological applications, since the external sources $s$ and $p$ are not realized in nature, and because moving away from the chiral limit in the $u$, $d$ sector produces only a small effect. In other words, when we include all terms proportional to $M^2 = B(m_u + m_d)$ we only add small corrections to a momentum structure that is already present at $O(p^4)$, renormalizing the corresponding low–energy constants.

Restricting ourselves to this simplified situation and to processes with at most four pions or currents (with not more than one axial current), the number of phenomenologically relevant terms goes down to sixteen:

$$\begin{aligned}
P_1,\ P_2,\ P_3 &\ \text{ contributing to } \ \pi\pi \to \pi\pi, \\
P_{29}, \ldots, P_{33}, P_{50} &\ \text{ contributing to } \ \gamma\gamma \to \pi\pi, \\
P_{36},\ P_{37},\ P_{38} &\ \text{ contributing to } \ \tau \to 3\pi\nu_\tau, \\
P_{44},\ P_{50} &\ \text{ contributing to } \ \pi \to l\nu\gamma, \\
P_{51},\ P_{53} &\ \text{ contributing to } \ F_V^\pi(t), \\
P_{52} &\ \text{ contributing to } \ \pi \to l\nu\gamma^*.
\end{aligned}$$

This is indeed a more manageable number of coupling constants. Still, the real difficulty here is to relate different observables in a useful and practicable manner.

The lagrangean we have constructed shows that at $O(p^6)$ chiral symmetry becomes much less restrictive than in lower orders. In other words, the corresponding Ward identities involve a large number of different observables. Either one finds a way to calculate, or at least estimate, the various constants appearing at this order, or the use of this lagrangean for relating different observables via chiral symmetry will be rather limited.

## 5. Conclusion

We have constructed the most general even-intrinsic-parity lagrangean of order $p^6$ in the mesonic sector for the strong interaction in the presence of external vector, axial-vector, scalar and pseudoscalar fields. We used partial integration in the action, the equations of motion, or equivalently field redefinitions, and Bianchi identities in order



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to reduce the number of terms to a minimal set. We presented a general proof for the equivalence between EOM and field redefinitions for eliminating spurious terms in chiral lagrangeans.

The lagrangean contains 112 in principle measurable terms and 3 contact terms for the general case of $n$ light flavours. For $n = 3$ (2), the Cayley-Hamilton relations reduce the respective lagrangeans to 90 (53) measurable terms and 4 (4) contact terms. The differences between our result and earlier ones were discussed.

The chiral lagrangean of $O(p^6)$ is a necessary requisite for the renormalization program at the two-loop level [7, 8]. The low-energy constants of this lagrangean parametrize the most general local solution of $O(p^6)$ of the chiral Ward identities. Although it will not be possible to determine all renormalized low-energy constants by comparison with experiment, we now have a basis at our disposal for investigating those coupling constants with additional theoretical input beyond pure symmetry considerations (resonance saturation, large-$N_c$, lattice simulations, chiral models,...).

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## A. Field redefinitions and equations of motion

In this appendix we give a general proof for the equivalence of two procedures for removing terms in effective chiral lagrangeans:

1. using the lowest-order EOM;

2. performing field redefinitions (FR).

The following proof completes the discussion of ref. [12] where it was explicitly shown that EOM terms can be removed by FR.

In full generality, an element of the chiral coset space is of the form $(l_L(\phi), l_R(\phi))$. Under a chiral transformation $g = (g_L, g_R) \in \mathrm{SU}(n)_L \times \mathrm{SU}(n)_R$,

$$l_A(\phi) \to g_A l_A(\phi) h(g, \phi)^{-1}, \qquad A = L, R. \tag{A.1}$$

Parity as an automorphism of the chiral group interchanges $l_L$ and $l_R$. The matrix field $U(\phi)$ used in the LR-basis is defined as

$$U(\phi) = l_R(\phi) l_L(\phi)^\dagger . \tag{A.2}$$

The usual choice of coset coordinates [13] corresponds to

$$
\begin{aligned}
l_R(\phi) &= l_L(\phi)^\dagger =: u(\phi) \\
U(\phi) &= u(\phi)^2 \,.
\end{aligned}
\tag{A.3}
$$

Let us now assume that the general chiral lagrangean has been constructed in this standard basis including all EOM terms (external fields are denoted collectively as $j$):

$$
\mathcal{L}(\phi,j) = \mathcal{L}_2(\phi,j) + \sum_{n\geq 2} \Big( \mathcal{L}_{2n}(\phi,j) + \langle X_2(\phi,j)\mathcal{E}_{2n-2}(\phi,j)\rangle \Big) \,. \tag{A.4}
$$

The last terms are the EOM terms of chiral order $2n$ with

$$
X_2(\phi,j) = \nabla^\mu u_\mu - \frac{i}{2}\left(\chi_- - \frac{1}{n}\langle\chi_-\rangle\right) . \tag{A.5}
$$

We now perform a FR which amounts to a reparametrization of coset space. The most general such transformation is of the form

$$
\hat{l}_R(\phi,j) = u(\phi) e^{\frac{i}{2}\xi(\phi,j)} e^{i\sigma(\phi,j)} \,. \tag{A.6}
$$

Because $\hat{l}_R$ and $u$ are elements of SU$(n)$, the matrix fields $\xi$ and $\sigma$ are hermitian and traceless. We have split the transformation matrix into two parts distinguished by their intrinsic parity: $\sigma$ is even while $\xi$ is odd. Parity then fixes $\hat{l}_L$ to be

$$
\hat{l}_L(\phi,j) = u(\phi)^\dagger e^{-\frac{i}{2}\xi(\phi,j)} e^{i\sigma(\phi,j)} \,. \tag{A.7}
$$

For showing the equivalence between EOM and FR, it is important to realize that $\sigma(\phi,j)$ has no effect on the chiral lagrangean. This is most easily seen by looking at the matrix $\hat{U}$ after the FR:

$$
\hat{U} = \hat{l}_R \hat{l}_L^\dagger = u e^{\frac{i}{2}\xi} e^{i\sigma} e^{-i\sigma} e^{\frac{i}{2}\xi} u = u e^{i\xi} u \,. \tag{A.8}
$$

Therefore, the most general FR is given by a hermitian, traceless matrix field $\xi(\phi,j)$ of odd intrinsic parity. Moreover, from the definition (A.6) and the transformation property (A.1), the field $\xi(\phi,j)$ transforms as in (2.3). Since $X_2$ in (A.5) shares these properties, this also holds for the matrices $\mathcal{E}_{2n}(\phi,j)$ in (A.4). In particular, $\mathcal{E}_{2n}(\phi,j)$ can be taken to be traceless in all generality since the trace of $\mathcal{E}_{2n}(\phi,j)$ does not contribute in (A.4). Therefore, each term in $\mathcal{E}_{2n}(\phi,j)$ defines a possible FR via $\xi(\phi,j)$ and vice versa.

The remainder of the proof is as in ref. [12]. In a first step, we choose $\xi = \xi_2$ of $O(p^2)$ as

$$
\xi_2(\phi,j) = -\frac{2}{F^2}\mathcal{E}_2(\phi,j) \,. \tag{A.9}
$$

After this FR, the lowest-order lagrangean (2.5) turns into (including partial integration in the action)

$$\mathcal{L}_2 \to \mathcal{L}_2 + \frac{F^2}{2}\langle \xi_2 X_2 \rangle + O(p^6) = \mathcal{L}_2 - \langle X_2 \mathcal{E}_2 \rangle + O(p^6)\,, \tag{A.10}$$

canceling the EOM terms of $O(p^4)$ in (A.4). The chiral lagrangean of $O(p^6)$ and higher (including the EOM terms) will of course be modified. Since we are considering the most general chiral lagrangean in (A.4), only the coefficients of the higher-order terms may have changed, but not their structure.

We can now repeat the procedure by introducing a further FR

$$\xi_4(\phi, j) = -\frac{2}{F^2}\mathcal{E}_4(\phi, j) \tag{A.11}$$

that removes the EOM terms of $O(p^6)$ in (A.4) without modifying $\mathcal{L}_2$ and $\mathcal{L}_4$, but affecting of course the lagrangean of $O(p^8)$ and higher. Continuing this procedure, we can remove all EOM terms order by order in $p^2$ by successive FR of the form

$$\hat{l}_R(\phi, j) = u(\phi) e^{\frac{i}{2}\xi_2(\phi,j)} \dots e^{\frac{i}{2}\xi_{2n}(\phi,j)} \dots \tag{A.12}$$

Since the most general expression for $\mathcal{E}_{2n}(\phi, j)$ defines the most general possible FR in terms of $\xi_{2n}(\phi, j)$ and vice versa, the equivalence between using the EOM and performing FR is established.

## B. Cayley-Hamilton relations

The Cayley-Hamilton theorem for $n = 3$ can be used to eliminate the following 21 terms from the SU($n$) list. As emphasized in the introduction, we eliminate preferentially multiple trace terms (same for the $n = 2$ relations below) to make the large-$N_c$ structure manifest. Here as in the main text, $Y_i$ stands for the $i$-th monomial in the SU($n$) column of table 2. The left-hand side mentions in square brackets also the numbering of the monomial removed. We use the notation $u\cdot u = u_\rho u^\rho$.

$$\begin{aligned}
\langle h_{\mu\nu} u_\rho \rangle \langle h^{\mu\nu} u^\rho \rangle [Y_4] &= 2\,Y_1 - 1/2\,Y_2 + \,Y_3 \\
\langle h_{\mu\nu} u_\rho \rangle \langle h^{\mu\rho} u^\nu \rangle [Y_6] &= 2/3\,Y_1 - 2/3\,Y_3 + \,Y_5 + \,Y_{28} - 1/2\,Y_{29} - \\
&\quad -1/3\,Y_{30} - \,Y_{31} + 2/3\,Y_{33} - 2/3\,Y_{37} - 4/3\,Y_{49} + \\
&\quad +1/3\,Y_{50} - 2/3\,Y_{52} + 8/3\,Y_{54} - 1/3\,Y_{57} + 2/3\,Y_{58} - \\
&\quad -4/3\,Y_{60} - 2/3\,Y_{64} + 2/3\,Y_{65} - 4/3\,Y_{66} - 8/3\,Y_{67} + \\
&\quad +2/3\,Y_{68} + 4/3\,Y_{86} - 2/3\,Y_{87} - 4/3\,Y_{88} + 2/3\,Y_{89} - \\
&\quad -2/3\,Y_{90} + 2/3\,Y_{92} + 2\,Y_{94} - 2/3\,Y_{95} - 1/3\,Y_{96} - \\
&\quad -2/3\,Y_{97} + 1/3\,Y_{99} + 2/3\,Y_{105} + 1/3\,Y_{106} \\
\langle u\cdot u \rangle^2 \langle \chi_+ \rangle [Y_{10}] &= -4\,Y_7 + 2\,Y_8 + 3\,Y_9 - 2\,Y_{11} + 2\,Y_{12}
\end{aligned}$$

$$
\begin{aligned}
\langle \chi_+ u_\mu u_\nu \rangle \langle u^\mu u^\nu \rangle [Y_{15}] &= Y_7 - 1/2\, Y_9 + Y_{11} - Y_{12} + Y_{13} \\
\langle \chi_+ \rangle \langle u_\mu u_\nu \rangle^2 [Y_{16}] &= 2\, Y_7 + Y_8 - 3/2\, Y_9 + Y_{11} - Y_{12} + Y_{14} \\
\langle u\cdot u \rangle \langle \chi_+ \rangle^2 [Y_{22}] &= -4\, Y_{19} + 4\, Y_{20} + Y_{21} - 2\, Y_{23} + 2\, Y_{24} \\
i\langle h_{\mu\nu} u^\mu \rangle \langle \chi_- u^\nu \rangle [Y_{32}] &= Y_{28} - 1/2\, Y_{29} - Y_{30} + Y_{31} \\
\langle u\cdot u \rangle \langle \chi_- \rangle^2 [Y_{36}] &= -4\, Y_{33} + 4\, Y_{34} + Y_{35} - 2\, Y_{37} + 2\, Y_{38} \\
\langle u\cdot u \rangle^3 [Y_{51}] &= -4\, Y_{49} + 5\, Y_{50} - 2\, Y_{52} + 2\, Y_{53} \\
\langle u\cdot u u_\mu u_\nu \rangle \langle u^\mu u^\nu \rangle [Y_{55}] &= Y_{49} - 1/2\, Y_{50} + Y_{52} - Y_{53} + Y_{54} \\
\langle u\cdot u \rangle \langle u_\mu u_\nu \rangle^2 [Y_{56}] &= 2\, Y_{49} - 1/2\, Y_{50} + Y_{52} - Y_{53} + Y_{57} \\
\langle u_\mu u_\nu u_\rho \rangle^2 [Y_{59}] &= -\, Y_{49} + 3/4\, Y_{50} - 3/2\, Y_{52} + 3/2\, Y_{53} + Y_{58} \\
\langle u_\mu u_\nu u_\rho \rangle \langle u^\mu u^\rho u^\nu \rangle [Y_{61}] &= -\, Y_{49} + 1/4 Y_{50} + 1/2 Y_{52} + 3/2 Y_{53} - 2 Y_{54} + 1/2 Y_{57} + Y_{60} \\
\langle u_\mu u_\nu \rangle \langle u_\rho u^\mu u^\rho u^\nu \rangle [Y_{62}] &= -\, Y_{53} + 2\, Y_{54} - 1/2\, Y_{57} + Y_{60} \\
\langle u_\mu u_\nu \rangle \langle u^\mu u_\rho \rangle \langle u^\nu u^\rho \rangle [Y_{63}] &= Y_{49} - 3/4\, Y_{50} + 3/2\, Y_{52} - 5/2\, Y_{53} + 4\, Y_{54} - Y_{57} + Y_{60} \\
i\langle f_{+\mu\nu} u_\rho \rangle \langle u^\mu u^\nu u^\rho \rangle [Y_{69}] &= Y_{64} - 1/2\, Y_{65} + Y_{66} \\
i\langle f_{+\mu\nu} [u^\mu, u_\rho] \rangle \langle u^\nu u^\rho \rangle [Y_{70}] &= Y_{64} - Y_{65} + 2\, Y_{67} + Y_{68} \\
\langle f_{+\mu\nu} u_\rho \rangle^2 [Y_{74}] &= 2\, Y_{71} - 1/2\, Y_{72} + Y_{73} \\
\langle f_{+\mu\nu} u_\rho \rangle \langle f_+^{\mu\rho} u^\nu \rangle [Y_{79}] &= 2\, Y_{75} + 2\, Y_{76} - Y_{77} + Y_{78} - Y_{80} \\
\langle f_{-\mu\nu} u_\rho \rangle^2 [Y_{93}] &= 2\, Y_{90} - 1/2\, Y_{91} + Y_{92} \\
\langle f_{-\mu\nu} u_\rho \rangle \langle f_-^{\mu\rho} u^\nu \rangle [Y_{98}] &= 2\, Y_{94} + 2\, Y_{95} - Y_{96} + Y_{97} - Y_{99}\,. 
\end{aligned} \tag{B.1}
$$

All these relations can be derived from the identity (3.1) by multiplying with appropriate chiral matrices and taking traces. Only the second relation of (B.1) needs some more explanation. The original relation takes the simpler form

$$
Y_{86} - Y_{87} - Y_{88} + Y_{89} + \langle f_{-\mu\nu}(h^{\nu\rho} u^\mu u_\rho + u_\rho u^\mu h^{\nu\rho}) \rangle + \langle f_{-\mu\nu} u_\rho \rangle \langle h^{\nu\rho} u^\mu \rangle = 0\,. \tag{B.2}
$$

However, we have already replaced the last two terms in this relation in the SU($n$) basis by partial integration and EOM leading to the more complicated form of the second relation in (B.1).

In addition, the extra contact term has been used to remove $Y_{42}$ via (3.4).

As explained in sect. 2, there are 37 additional linear relations for $n = 2$ that can be written in the form

$$
\begin{aligned}
\langle u\cdot u \rangle \langle h_{\mu\nu} h^{\mu\nu} \rangle [Y_2] &= 2\, Y_1 \\
\langle (u\cdot u)^2 \rangle \langle \chi_+ \rangle [Y_8] &= 2\, Y_7 \\
\langle u\cdot u \rangle \langle u\cdot u \chi_+ \rangle [Y_9] &= 2\, Y_7 \\
\langle u\cdot u u_\mu \chi_+ u^\mu \rangle [Y_{11}] &= Y_7 \\
\langle u\cdot u u_\mu \rangle \langle \chi_+ u^\mu \rangle [Y_{12}] &= 0 \\
\langle \chi_+ \rangle \langle u_\mu u_\nu u^\mu u^\nu \rangle [Y_{14}] &= 2\, Y_{13}
\end{aligned}
$$

$$
\begin{aligned}
\langle\chi_+\rangle\langle h_{\mu\nu}h^{\mu\nu}\rangle[Y_{18}] &= 2\,Y_{17}\\
\langle u\cdot u\rangle\langle\chi_+^2\rangle[Y_{21}] &= 2\,Y_{19}\\
\langle\chi_+u_\mu\rangle^2[Y_{24}] &= \ Y_{19} - \ Y_{20} + \ Y_{23}\\
\langle\chi_+\rangle^3[Y_{27}] &= -2\,Y_{25} + 3\,Y_{26}\\
i\langle\chi_-h_{\mu\nu}\rangle\langle u^\mu u^\nu\rangle[Y_{29}] &= \ Y_{28}\\
i\langle h_{\mu\nu}u^\mu u^\nu\rangle\langle\chi_-\rangle[Y_{30}] &= 0\\
\langle u\cdot u\rangle\langle\chi_-^2\rangle[Y_{35}] &= 2\,Y_{33}\\
\langle u_\mu\chi_-\rangle^2[Y_{38}] &= \ Y_{33} - \ Y_{34} + \ Y_{37}\\
\langle\chi_+\rangle\langle\chi_-\rangle^2[Y_{42}] &= -2\,Y_{39} + \ Y_{40} + 2\,Y_{41}\\
i\langle\chi_{+\mu}\rangle\langle\chi_-u^\mu\rangle[Y_{45}] &= \ Y_{43} - \ Y_{44}\\
\langle(u\cdot u)^2\rangle\langle u\cdot u\rangle[Y_{50}] &= 2\,Y_{49}\\
\langle u\cdot uu_\mu u\cdot uu^\mu\rangle[Y_{52}] &= \ Y_{49}\\
\langle u\cdot uu_\mu\rangle^2[Y_{53}] &= 0\\
\langle u\cdot u\rangle\langle u_\mu u_\nu u^\mu u^\nu\rangle[Y_{57}] &= 2\,Y_{54}\\
\langle u_\mu u_\nu u_\rho u^\mu u^\rho u^\nu\rangle[Y_{60}] &= \ Y_{49} + \ Y_{54} - \ Y_{58}\\
i\langle f_{+\mu\nu}\{u\cdot u, u^\mu u^\nu\}\rangle[Y_{64}] &= 2\,Y_{67}\\
i\langle u\cdot u\rangle\langle f_{+\mu\nu}u^\mu u^\nu\rangle[Y_{65}] &= 2\,Y_{67}\\
i\langle f_{+\mu\nu}\{u_\rho, u^\mu u^\rho u^\nu\}\rangle[Y_{68}] &= -\,Y_{66} - \ Y_{67}\\
\langle u\cdot u\rangle\langle f_{+\mu\nu}f_+^{\mu\nu}\rangle[Y_{72}] &= 2\,Y_{71}\\
\langle f_{+\mu\nu}f_+^{\mu\rho}\rangle\langle u^\nu u_\rho\rangle[Y_{77}] &= \ Y_{75} + Y_{76}\\
\langle f_{+\mu\nu}u^\nu\rangle\langle f_+^{\mu\rho}u_\rho\rangle[Y_{80}] &= Y_{76} + 1/2\,Y_{78}\\
\langle\chi_+\rangle\langle f_{+\mu\nu}f_+^{\mu\nu}\rangle[Y_{82}] &= 2\,Y_{81}\\
i\langle f_{+\mu\nu}\{\chi_+, u^\mu u^\nu\}\rangle[Y_{83}] &= 2\,Y_{85}\\
i\langle\chi_+\rangle\langle f_{+\mu\nu}u^\mu u^\nu\rangle[Y_{84}] &= 2\,Y_{85}\\
\langle f_{-\mu\nu}u^\mu\rangle\langle h^{\nu\rho}u_\rho\rangle[Y_{88}] &= 1/2\,Y_{86} + 1/2\,Y_{89}\\
\langle u\cdot u\rangle\langle f_{-\mu\nu}f_-^{\mu\nu}\rangle[Y_{91}] &= 2\,Y_{90}\\
\langle f_{-\mu\nu}f_-^{\mu\rho}\rangle\langle u^\nu u_\rho\rangle[Y_{96}] &= Y_{94} + \ Y_{95}\\
\langle f_{-\mu\nu}u^\nu\rangle\langle f_-^{\mu\rho}u_\rho\rangle[Y_{99}] &= \ Y_{95} + 1/2\,Y_{97}\\
\langle\chi_+\rangle\langle f_{-\mu\nu}f_-^{\mu\nu}\rangle[Y_{103}] &= 2\,Y_{102}\\
i\langle f_{-\mu\nu}u^\nu\rangle\langle u^\mu\chi_-\rangle[Y_{106}] &= \ Y_{105}/2\\
\langle\chi_+^\mu\rangle\langle f_{-\mu\nu}u^\nu\rangle[Y_{108}] &= \ Y_{107}\,.
\end{aligned}
\tag{B.3}
$$

In addition, the extra contact term has been used to remove $Y_{46}$ via (3.5).

## C. Comparison with the lagrangean of Fearing and Scherer

Fearing and Scherer (FS) [9] have constructed chiral lagrangeans of $O(p^6)$ in the LR-basis, but otherwise they have followed a strategy very similar to ours. We have explicitly checked that all their terms can be expressed in our basis of independent chiral invariants of $O(p^6)$. However, since they have[5] 129 instead of our 112+3 monomials for SU($n$) and 111 vs. 90+4 for SU(3), their basis can not be minimal.

Two obvious reasons for these differences are the Bianchi identity and the contact terms, both of which FS did not take into consideration. Nevertheless, this does not fully account for the different number of terms, which can then only be due to partial integration and/or EOM for general $n$ and to the Cayley-Hamilton theorem for $n = 3$.

We now exhibit some examples of linear dependences in the FS lagrangean. Since we do not intend to provide a full translation of their 129 (111) terms into our basis, we restrict the comparison to the case where all external fields are set to zero. In our basis, this leaves 21 terms denoted $Y_1, \ldots, Y_6$ and $Y_{49}, \ldots, Y_{63}$ for chiral SU($n$). In comparison, FS also have the equivalent of $Y_{49}, \ldots, Y_{63}$, but 8 additional terms instead of our $Y_1, \ldots, Y_6$. Four of their additional terms are directly related to $Y_1, Y_2, Y_4, Y_6$ (and to $Y_{49}, \ldots, Y_{63}$). The other four (in our notation and with all external fields set to zero) can be expressed in our basis via partial integration and application of the EOM:

$$\begin{aligned}
\langle h_{\mu\nu} h^{\mu\rho} u^\nu u_\rho \rangle &= -1/2\, Y_3 + Y_{54} - Y_{60} \\
\langle h_{\mu\nu} h^{\mu\rho} u_\rho u^\nu \rangle &= -Y_1 - 1/2\, Y_5 + Y_{49} + Y_{52} - 2\, Y_{54} \\
\langle h_{\mu\nu} h^{\mu\rho} \rangle \langle u^\nu u_\rho \rangle &= -Y_4 - Y_6 + 2\, Y_{55} - 2\, Y_{62} \\
\langle h_{\mu\nu} u^\nu \rangle \langle h^{\mu\rho} u_\rho \rangle &= -1/2\, Y_2 + Y_{50} - Y_{57}\,.
\end{aligned} \tag{C.1}$$

Thus, for general $n$ there are two linear relations among the 23 terms used by FS.

Turning now to SU(3), FS have 12 terms with six powers of $u_\mu$ while we have only eight. We have six of the $Y_i$ in common (i=49,50,52,53,54,58). The remaining six invariants of FS can be expressed with the help of Cayley-Hamilton in terms of our eight invariants that also include $Y_{57}$ and $Y_{60}$. The following equalities already appear in (B.1).

$$\begin{aligned}
\langle u\cdot u \rangle^3 &= -4\, Y_{49} + 5\, Y_{50} - 2\, Y_{52} + 2\, Y_{53} \qquad \text{(C.2)}\\
\langle u\cdot u u_\mu u_\nu \rangle \langle u^\mu u^\nu \rangle &= Y_{49} - 1/2\, Y_{50} + Y_{52} - Y_{53} + Y_{54} \\
\langle u\cdot u \rangle \langle u_\mu u_\nu \rangle^2 &= 2\, Y_{49} - 1/2\, Y_{50} + Y_{52} - Y_{53} + Y_{57} \\
\langle u_\mu u_\nu u_\rho \rangle^2 &= -Y_{49} + 3/4\, Y_{50} - 3/2\, Y_{52} + 3/2\, Y_{53} + Y_{58} \\
\langle u_\mu u_\nu u_\rho \rangle \langle u^\mu u^\rho u^\nu \rangle &= -Y_{49} + 1/4\, Y_{50} + 1/2\, Y_{52} + 3/2\, Y_{53} -
\end{aligned}$$

[5] In [9] no attempt was made to create a minimal basis for the $n$-flavour case. They quote 18 trace relations and 111 terms for 3 flavours, hence we use 129 as their number of terms for $n$ flavours.



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$$-2\,Y_{54} + 1/2\,Y_{57} + \ Y_{60}$$

$$\langle u_\mu u_\nu\rangle\langle u^\mu u_\rho\rangle\langle u^\nu u^\rho\rangle \ = \ Y_{49} - 3/4\,Y_{50} + 3/2\,Y_{52} - 5/2\,Y_{53} + 4\,Y_{54} - \ Y_{57} + \ Y_{60}\,.$$

Consequently, there are four linear relations of the Cayley-Hamilton type among the 12 terms of FS.

| monomial ($Y_i$) | SU(n) | SU(3) | SU(2) | contributes to |
|---|---|---|---|---|
| $\langle u\cdot u h_{\mu\nu}h^{\mu\nu}\rangle$ | 1 | 1 | 1 | $\pi\pi\to\pi\pi$ |
| $\langle u\cdot u\rangle\langle h_{\mu\nu}h^{\mu\nu}\rangle$ | 2 | 2 | | $\pi\pi\to\pi\pi$ |
| $\langle h_{\mu\nu}u_\rho h^{\mu\nu}u^\rho\rangle$ | 3 | 3 | 2 | $\pi\pi\to\pi\pi$ |
| $\langle h_{\mu\nu}u_\rho\rangle\langle h^{\mu\nu}u^\rho\rangle$ | 4 | | | $\pi\pi\to\pi\pi$ |
| $\langle h_{\mu\nu}\left(u_\rho h^{\mu\rho}u^\nu + u^\nu h^{\mu\rho}u_\rho\right)\rangle$ | 5 | 4 | 3 | $\pi\pi\to\pi\pi$ |
| $\langle h_{\mu\nu}u_\rho\rangle\langle h^{\mu\rho}u^\nu\rangle$ | 6 | | | $\pi\pi\to\pi\pi$ |
| | | | | |
| $\langle (u\cdot u)^2\chi_+\rangle$ | 7 | 5 | 4 | $\pi\pi\to\pi\pi$ |
| $\langle (u\cdot u)^2\rangle\langle\chi_+\rangle$ | 8 | 6 | | $\pi\pi\to\pi\pi$ |
| $\langle u\cdot u\rangle\langle u\cdot u\chi_+\rangle$ | 9 | 7 | | $\pi\pi\to\pi\pi$ |
| $\langle u\cdot u\rangle^2\langle\chi_+\rangle$ | 10 | | | $\pi\pi\to\pi\pi$ |
| $\langle u\cdot u u_\mu\chi_+u^\mu\rangle$ | 11 | 8 | | $\pi\pi\to\pi\pi$ |
| $\langle u\cdot u u_\mu\rangle\langle\chi_+u^\mu\rangle$ | 12 | 9 | | $\pi\pi\to\pi\pi$ |
| $\langle \chi_+u_\mu u_\nu u^\mu u^\nu\rangle$ | 13 | 10 | 5 | $\pi\pi\to\pi\pi$ |
| $\langle\chi_+\rangle\langle u_\mu u_\nu u^\mu u^\nu\rangle$ | 14 | 11 | | $\pi\pi\to\pi\pi$ |
| $\langle\chi_+u_\mu u_\nu\rangle\langle u^\mu u^\nu\rangle$ | 15 | | | $\pi\pi\to\pi\pi$ |
| $\langle\chi_+\rangle\langle u_\mu u_\nu\rangle^2$ | 16 | | | $\pi\pi\to\pi\pi$ |
| $\langle\chi_+h_{\mu\nu}h^{\mu\nu}\rangle$ | 17 | 12 | 6 | $\langle\pi\pi\rangle$ |
| $\langle\chi_+\rangle\langle h_{\mu\nu}h^{\mu\nu}\rangle$ | 18 | 13 | | $\langle\pi\pi\rangle$ |
| $\langle u\cdot u\chi_+^2\rangle$ | 19 | 14 | 7 | $\langle\pi\pi\rangle$ |
| $\langle u\cdot u\chi_+\rangle\langle\chi_+\rangle$ | 20 | 15 | 8 | $\langle\pi\pi\rangle$ |
| $\langle u\cdot u\rangle\langle\chi_+^2\rangle$ | 21 | 16 | | $\langle\pi\pi\rangle$ |
| $\langle u\cdot u\rangle\langle\chi_+\rangle^2$ | 22 | | | $\langle\pi\pi\rangle$ |
| $\langle\chi_+u_\mu\chi_+u^\mu\rangle$ | 23 | 17 | 9 | $\langle\pi\pi\rangle$ |
| $\langle\chi_+u_\mu\rangle^2$ | 24 | 18 | | $\langle\pi\pi\rangle$ |
| $\langle\chi_+^3\rangle$ | 25 | 19 | 10 | $\langle\pi\pi\rangle$ |
| $\langle\chi_+^2\rangle\langle\chi_+\rangle$ | 26 | 20 | 11 | $\langle\pi\pi\rangle$ |
| $\langle\chi_+\rangle^3$ | 27 | 21 | | $\langle\pi\pi\rangle$ |

**Table 2**

| monomial ($Y_i$) | SU(n) | SU(3) | SU(2) | contributes to |
|---|---|---|---|---|
| i $\langle \chi_- \{h_{\mu\nu}, u^\mu u^\nu\}\rangle$ | 28 | 22 | 12 | $\pi\pi \to \pi\pi$ |
| i $\langle \chi_- h_{\mu\nu}\rangle\langle u^\mu u^\nu\rangle$ | 29 | 23 | | $\pi\pi \to \pi\pi$ |
| i $\langle h_{\mu\nu} u^\mu u^\nu\rangle\langle \chi_-\rangle$ | 30 | 24 | | $\pi\pi \to \pi\pi$ |
| i $\langle h_{\mu\nu} u^\mu \chi_- u^\nu\rangle$ | 31 | 25 | 13 | $\pi\pi \to \pi\pi$ |
| i $\langle h_{\mu\nu} u^\mu\rangle\langle \chi_- u^\nu\rangle$ | 32 | | | $\pi\pi \to \pi\pi$ |
| $\langle u\cdot u \chi_-^2\rangle$ | 33 | 26 | 14 | $\pi\pi \to \pi\pi$ |
| $\langle u\cdot u \chi_-\rangle\langle \chi_-\rangle$ | 34 | 27 | 15 | $\pi\pi \to \pi\pi$ |
| $\langle u\cdot u\rangle\langle \chi_-^2\rangle$ | 35 | 28 | | $\pi\pi \to \pi\pi$ |
| $\langle u\cdot u\rangle\langle \chi_-\rangle^2$ | 36 | | | $\pi\pi \to \pi\pi$ |
| $\langle u_\mu \chi_- u^\mu \chi_-\rangle$ | 37 | 29 | 16 | $\pi\pi \to \pi\pi$ |
| $\langle u_\mu \chi_-\rangle^2$ | 38 | 30 | | $\pi\pi \to \pi\pi$ |
| $\langle \chi_-^2 \chi_+\rangle$ | 39 | 31 | 17 | $\langle\pi\pi\rangle$ |
| $\langle \chi_+\rangle\langle \chi_-^2\rangle$ | 40 | 32 | 18 | $\langle\pi\pi\rangle$ |
| $\langle \chi_+ \chi_-\rangle\langle \chi_-\rangle$ | 41 | 33 | 19 | $\langle\pi\pi\rangle$ |
| $\langle \chi_+\rangle\langle \chi_-\rangle^2$ | 42 | | | $\langle\pi\pi\rangle$ |
| i $\langle \chi_- \{\chi_{+\mu}, u^\mu\}\rangle$ | 43 | 34 | 20 | $F_S^\pi(t)$ |
| i $\langle \chi_-\rangle\langle \chi_{+\mu} u^\mu\rangle$ | 44 | 35 | 21 | $F_S^\pi(t)$ |
| i $\langle \chi_{+\mu}\rangle\langle \chi_- u^\mu\rangle$ | 45 | 36 | | $F_S^\pi(t)$ |
| $\langle \chi_{-\mu}\rangle^2$ | 46 | 37 | | $\langle SS\rangle$ |
| $\langle \chi_{+\mu} \chi_+^\mu\rangle$ | 47 | 38 | 22 | $\langle SS\rangle$ |
| $\langle \chi_{+\mu}\rangle^2$ | 48 | 39 | 23 | $\langle SS\rangle$ |
| | | | | |
| $\langle (u\cdot u)^3\rangle$ | 49 | 40 | 24 | $\pi\pi \to 4\pi$ |
| $\langle (u\cdot u)^2\rangle\langle u\cdot u\rangle$ | 50 | 41 | | $\pi\pi \to 4\pi$ |
| $\langle u\cdot u\rangle^3$ | 51 | | | $\pi\pi \to 4\pi$ |
| $\langle u\cdot u u_\mu u\cdot u u^\mu\rangle$ | 52 | 42 | | $\pi\pi \to 4\pi$ |
| $\langle u\cdot u u_\mu\rangle^2$ | 53 | 43 | | $\pi\pi \to 4\pi$ |
| $\langle u\cdot u u_\mu u_\nu u^\mu u^\nu\rangle$ | 54 | 44 | 25 | $\pi\pi \to 4\pi$ |
| $\langle u\cdot u u_\mu u_\nu\rangle\langle u^\mu u^\nu\rangle$ | 55 | | | $\pi\pi \to 4\pi$ |
| $\langle u\cdot u\rangle\langle u_\mu u_\nu\rangle^2$ | 56 | | | $\pi\pi \to 4\pi$ |
| $\langle u\cdot u\rangle\langle u_\mu u_\nu u^\mu u^\nu\rangle$ | 57 | 45 | | $\pi\pi \to 4\pi$ |
| $\langle u_\mu u_\nu u_\rho u^\mu u^\nu u^\rho\rangle$ | 58 | 46 | 26 | $\pi\pi \to 4\pi$ |
| $\langle u_\mu u_\nu u_\rho\rangle^2$ | 59 | | | $\pi\pi \to 4\pi$ |

**Table 2**

| monomial ($Y_i$) | SU(n) | SU(3) | SU(2) | contributes to |
|---|---|---|---|---|
| $\langle u_\mu u_\nu u_\rho u^\mu u^\rho u^\nu \rangle$ | 60 | 47 | | $\pi\pi \to 4\pi$ |
| $\langle u_\mu u_\nu u_\rho \rangle \langle u^\mu u^\rho u^\nu \rangle$ | 61 | | | $\pi\pi \to 4\pi$ |
| $\langle u_\mu u_\nu \rangle \langle u_\rho u^\mu u^\rho u^\nu \rangle$ | 62 | | | $\pi\pi \to 4\pi$ |
| $\langle u_\mu u_\nu \rangle \langle u^\mu u_\rho \rangle \langle u^\nu u^\rho \rangle$ | 63 | | | $\pi\pi \to 4\pi$ |
| | | | | |
| i $\langle f_{+\mu\nu} \{u\cdot u, u^\mu u^\nu\} \rangle$ | 64 | 48 | | $\gamma^* \to 4\pi$ |
| i $\langle u\cdot u \rangle \langle f_{+\mu\nu} u^\mu u^\nu \rangle$ | 65 | 49 | | $\gamma^* \to 4\pi$ |
| i $\langle f_{+\mu\nu} u_\rho u^\mu u^\nu u^\rho \rangle$ | 66 | 50 | 27 | $\gamma^* \to 4\pi$ |
| i $\langle f_{+\mu\nu} u^\mu u\cdot u u^\nu \rangle$ | 67 | 51 | 28 | $\gamma^* \to 4\pi$ |
| i $\langle f_{+\mu\nu} \{u_\rho, u^\mu u^\rho u^\nu\} \rangle$ | 68 | 52 | | $\gamma^* \to 4\pi$ |
| i $\langle f_{+\mu\nu} u_\rho \rangle \langle u^\mu u^\nu u^\rho \rangle$ | 69 | | | $\gamma^* \to 4\pi$ |
| i $\langle f_{+\mu\nu} [u^\mu, u_\rho] \rangle \langle u^\nu u^\rho \rangle$ | 70 | | | $\gamma^* \to 4\pi$ |
| $\langle u\cdot u f_{+\mu\nu} f_+^{\mu\nu} \rangle$ | 71 | 53 | 29 | $\gamma\gamma \to \pi\pi$ |
| $\langle u\cdot u \rangle \langle f_{+\mu\nu} f_+^{\mu\nu} \rangle$ | 72 | 54 | | $\gamma\gamma \to \pi\pi$ |
| $\langle f_{+\mu\nu} u_\rho f_+^{\mu\nu} u^\rho \rangle$ | 73 | 55 | 30 | $\gamma\gamma \to \pi\pi$ |
| $\langle f_{+\mu\nu} u_\rho \rangle^2$ | 74 | | | $\gamma\gamma \to \pi\pi$ |
| $\langle f_{+\mu\nu} f_+^{\mu\rho} u^\nu u_\rho \rangle$ | 75 | 56 | 31 | $\gamma\gamma \to \pi\pi$ |
| $\langle f_{+\mu\nu} f_+^{\mu\rho} u_\rho u^\nu \rangle$ | 76 | 57 | 32 | $\gamma\gamma \to \pi\pi$ |
| $\langle f_{+\mu\nu} f_+^{\mu\rho} \rangle \langle u^\nu u_\rho \rangle$ | 77 | 58 | | $\gamma\gamma \to \pi\pi$ |
| $\langle f_{+\mu\nu} \left( u_\rho f_+^{\mu\rho} u^\nu + u^\nu f_+^{\mu\rho} u_\rho \right) \rangle$ | 78 | 59 | 33 | $\gamma\gamma \to \pi\pi$ |
| $\langle f_{+\mu\nu} u_\rho \rangle \langle f_+^{\mu\rho} u^\nu \rangle$ | 79 | | | $\gamma\gamma \to \pi\pi$ |
| $\langle f_{+\mu\nu} u^\nu \rangle \langle f_+^{\mu\rho} u_\rho \rangle$ | 80 | 60 | | $\gamma\gamma \to \pi\pi$ |
| $\langle \chi_+ f_{+\mu\nu} f_+^{\mu\nu} \rangle$ | 81 | 61 | 34 | $\langle VV \rangle$ |
| $\langle \chi_+ \rangle \langle f_{+\mu\nu} f_+^{\mu\nu} \rangle$ | 82 | 62 | | $\langle VV \rangle$ |
| i $\langle f_{+\mu\nu} \{\chi_+, u^\mu u^\nu\} \rangle$ | 83 | 63 | | $F_V^\pi(t),\ K_{l3}$ |
| i $\langle \chi_+ \rangle \langle f_{+\mu\nu} u^\mu u^\nu \rangle$ | 84 | 64 | | $F_V^\pi(t),\ K_{l3}$ |
| i $\langle f_{+\mu\nu} u^\mu \chi_+ u^\nu \rangle$ | 85 | 65 | 35 | $F_V^\pi(t),\ K_{l3}$ |
| | | | | |
| $\langle f_{-\mu\nu} \left( h^{\nu\rho} u_\rho u^\mu + u^\mu u_\rho h^{\nu\rho} \right) \rangle$ | 86 | 66 | 36 | $K_{l4}$ |
| $\langle f_{-\mu\nu} h^{\nu\rho} \rangle \langle u^\mu u_\rho \rangle$ | 87 | 67 | 37 | $K_{l4}$ |
| $\langle f_{-\mu\nu} u^\mu \rangle \langle h^{\nu\rho} u_\rho \rangle$ | 88 | 68 | | $K_{l4}$ |
| $\langle f_{-\mu\nu} \left( u^\mu h^{\nu\rho} u_\rho + u_\rho h^{\nu\rho} u^\mu \right) \rangle$ | 89 | 69 | 38 | $K_{l4}$ |
| $\langle u\cdot u f_{-\mu\nu} f_-^{\mu\nu} \rangle$ | 90 | 70 | 39 | $K_{l4\gamma}$ |

**Table 2**

| monomial ($Y_i$) | SU(n) | SU(3) | SU(2) | contributes to |
|---|---|---|---|---|
| $\langle u\cdot u\rangle\langle f_{-\mu\nu}f_-^{\mu\nu}\rangle$ | 91 | 71 | | $K_{l4\gamma}$ |
| $\langle f_{-\mu\nu}u_\rho f_-^{\mu\nu}u^\rho\rangle$ | 92 | 72 | 40 | $K_{l4\gamma}$ |
| $\langle f_{-\mu\nu}u_\rho\rangle^2$ | 93 | | | $K_{l4\gamma}$ |
| $\langle f_{-\mu\nu}f_-^{\mu\rho}u^\nu u_\rho\rangle$ | 94 | 73 | 41 | $K_{l4\gamma}$ |
| $\langle f_{-\mu\nu}f_-^{\mu\rho}u_\rho u^\nu\rangle$ | 95 | 74 | 42 | $K_{l4\gamma}$ |
| $\langle f_{-\mu\nu}f_-^{\mu\rho}\rangle\langle u^\nu u_\rho\rangle$ | 96 | 75 | | $K_{l4\gamma}$ |
| $\langle f_{-\mu\nu}\left(u_\rho f_-^{\mu\rho}u^\nu + u^\nu f_-^{\mu\rho}u_\rho\right)\rangle$ | 97 | 76 | 43 | $K_{l4\gamma}$ |
| $\langle f_{-\mu\nu}u_\rho\rangle\langle f_-^{\mu\rho}u^\nu\rangle$ | 98 | | | $K_{l4\gamma}$ |
| $\langle f_{-\mu\nu}u^\nu\rangle\langle f_-^{\mu\rho}u_\rho\rangle$ | 99 | 77 | | $K_{l4\gamma}$ |
| $\mathrm{i}\,\langle f_{+\mu\nu}[f_-^{\nu\rho},h_\rho^\mu]\rangle$ | 100 | 78 | 44 | $\pi\to l\nu\gamma$ |
| $\mathrm{i}\,\langle f_{+\mu\nu}[f_-^{\nu\rho},f_{-\rho}^{\mu}]\rangle$ | 101 | 79 | 45 | $\langle VAA\rangle$ |
| $\langle\chi_+f_{-\mu\nu}f_-^{\mu\nu}\rangle$ | 102 | 80 | 46 | $\langle AA\rangle$ |
| $\langle\chi_+\rangle\langle f_{-\mu\nu}f_-^{\mu\nu}\rangle$ | 103 | 81 | | $\langle AA\rangle$ |
| $\langle f_{+\mu\nu}[f_-^{\mu\nu},\chi_-]\rangle$ | 104 | 82 | 47 | $\pi\to l\nu\gamma$ |
| $\mathrm{i}\,\langle f_{-\mu\nu}[\chi_-,u^\mu u^\nu]\rangle$ | 105 | 83 | 48 | $K_{l4}$ |
| $\mathrm{i}\,\langle f_{-\mu\nu}u^\nu\rangle\langle u^\mu\chi_-\rangle$ | 106 | 84 | | $K_{l4}$ |
| $\langle f_{-\mu\nu}\{\chi_+^\mu,u^\nu\}\rangle$ | 107 | 85 | 49 | $\langle VAA\rangle$ |
| $\langle\chi_+^\mu\rangle\langle f_{-\mu\nu}u^\nu\rangle$ | 108 | 86 | | $\langle VAA\rangle$ |
| $\langle\nabla_\rho f_{-\mu\nu}\nabla^\rho f_-^{\mu\nu}\rangle$ | 109 | 87 | 50 | $\langle AA\rangle$ |
| $\mathrm{i}\,\langle\nabla_\rho f_{+\mu\nu}[h^{\mu\rho},u^\nu]\rangle$ | 110 | 88 | 51 | $F_V^\pi(t),\ K_{l3}$ |
| $\mathrm{i}\,\langle\nabla^\mu f_{+\mu\nu}[f_-^{\nu\rho},u_\rho]\rangle$ | 111 | 89 | 52 | $\pi\to l\nu\gamma^*$ |
| $\mathrm{i}\,\langle\nabla^\mu f_{+\mu\nu}[h^{\nu\rho},u_\rho]\rangle$ | 112 | 90 | 53 | $F_V^\pi(t),\ K_{l3}$ |
| contact terms | | | | |
| $\langle D_\mu\chi D^\mu\chi^\dagger\rangle$ | 113 | 91 | 54 | |
| $\mathrm{i}\,\langle F_{L\mu\nu}F_L^{\mu\rho}F_{L\rho}^{\nu}\rangle + L\to R$ | 114 | 92 | 55 | |
| $\langle D_\rho F_{L\mu\nu}D^\rho F_L^{\mu\nu}\rangle + L\to R$ | 115 | 93 | 56 | |
| additional contact term for SU(3) | | | | |
| $\det(\chi)$ + h.c. | | 94 | | |
| additional contact term for SU(2) | | | | |
| $\langle D_\mu\chi D^\mu\widetilde{\chi}\rangle$ + h.c. | | | 57 | |

**Table 2:** Independent monomials of $O(p^6)$ for SU($n$), SU(3) and SU(2). The flavour trace is denoted by $\langle\ldots\rangle$. The first column lists the structure and the following three the numbering of independent terms for a number of flavours $n$, 3 and 2. The last column indicates the simplest quantity or process to which the term contributes.

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